

## PARTICLE TUNNELING AND SCATTERING IN A SPHERICAL THREE-DIMENSIONAL POTENTIAL WITH A BARRIER

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The particle tunneling and scattering through a three-dimensional rectangular barrier, containing a rectangular potential well inside, had been studied. The multiple internal reflections inside the well had been explicitly taken into account. The explicit analytical expressions between the *S*-matrix of elastic scattering and all probability amplitudes (*external and internal reflections, tunneling inside and tunneling outside*) had been obtained. All correspondent phase times had been studied. The perspectives of generalizing for the cases of the Coulombian barriers and nonrectangular potential wells are discussed.

### 1. Introduction

The time-dependent aspects in the process of the quantum tunneling of particles through 1-dimensional barriers were the object of several papers, reported, for instance, in [1] and [2] (and in all references quoted therein). On the other hand, the three-dimensional approach to this same problem is present in the literature with a less complete analysis. In fact, particle tunneling through a three dimensional barrier has been studied in a simplified way in the framework of the WKB approximation (see, for example [3 - 6]), or using only some elements of the time-dependent description in applications for some concrete tasks such as  $\alpha$ -decay. In this paper, we intend to study non-relativistic particle tunneling through a three-dimensional potential barrier with a spherical symmetry.

Initially (Sec. 2) we present usual general wave-packet and stationary description of particle elastic scattering in a spherically symmetric potential (a rectangular well surrounded by a rectangular barrier).

Then we will consider, in the central part of the system (Sec. 3), the impact of the particles with this potential as a sequence of two successive processes: in the first stage we think to an ingoing spherical ( $l = 0$ ) wave packet impinging from outside on the barrier, producing a reflected wave in the external region, tunneling through the barrier, and finally penetrating in the well where it is represented by an ingoing mode. In the second phase, we will consider the connection of the tunneling and reflections amplitudes with the *S*-matrix and the presence of an outgoing spherical wave from the well, which, after multiple reflections against the internal side of the barrier, tunnels through the barrier and produces, finally, an outgoing mode in the external region.

Finally (Sec. 4), the further perspectives and conclusions will be presented.

### 2. General wave-packet and stationary description

We shall refer to the various regions in this way: region (I) with  $r > R_2$  represents the external region of null potential, (II) delimited by  $R_1$  and  $R_2$  is the barrier region and the internal region (III) with  $r < R_1$  is the well (a section of the potential along the  $r$ -axis is shown in Fig. 1).

The particle time-dependent wave packet will be described by expression

$$\Psi(\vec{k}, \vec{r}, t) = \int d\vec{k}' g(\vec{k}' - \vec{k}) \psi(\vec{k}', \vec{r}) \exp(-iEt/\hbar) \quad (1)$$

with the stationary wave function

$$\psi(\vec{k}, \vec{r}) = \sum_l (2l+1) i^l \psi_l(k, r) P_l(\cos\theta) \quad (2)$$

( $k$  and  $E = \hbar^2 k^2 / 2m$  are the wave number and the kinetic energy, respectively,  $l = 0, 1, 2, \dots$  is the orbital quantum number,  $\theta$  is the angle between the directions of momentum-vector  $\vec{k}$  and radius-vector  $\vec{r}$ ,  $|\vec{k}'| = |\vec{k}|$ ,  $g(\vec{k}' - \vec{k})$  is a normalized amplitude weight factor), corresponding to the initial stationary plane wave

$$\exp(ikr\cos\theta) = \sum_l (2l+1) i^l j_l(kr) P_l(\cos\theta), \quad (3)$$

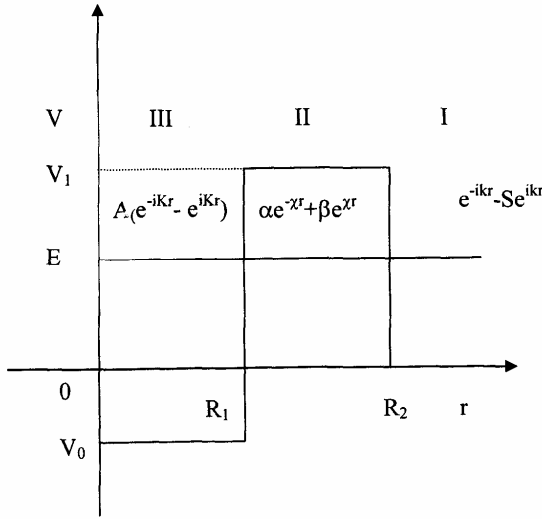


Fig. 1. Schematic stationary description of particle scattering in the presence of a rectangular barrier ( $l = 0$ ).

Now we pass to the  $s$ -scattering, i.e. to the case  $l = 0$ , introducing the functions

$$u_n(k, r) \equiv -2iq_n r \psi_{n,0}(k, r), \quad n = I, II, III, \quad q_I = k, \quad q_{II} = i\chi, \quad q_{III} = K, \quad (5)$$

namely

$$\begin{aligned} u_I(k, r) &= e^{-ikr} - S e^{ikr}, \quad S \equiv S_0 \\ u_{II}(k, r) &= \alpha e^{-\chi r} + \beta e^{\chi r}, \quad \alpha \equiv \alpha_0, \quad \beta \equiv \beta_0, \\ u_{III}(k, r) &= A(e^{-iKr} - e^{iKr}), \quad A \equiv A_0. \end{aligned} \quad (6)$$

Imposing the continuity conditions we have:

$$\begin{aligned} S &= e^{-2ikR_2} \frac{(K \cos KR_1 + \chi \sin KR_1)(\chi + ik) + e^{2\chi(R_1 - R_2)}(K \cos KR_1 - \chi \sin KR_1)(\chi - ik)}{(K \cos KR_1 + \chi \sin KR_1)(\chi - ik) + e^{2\chi(R_1 - R_2)}(K \cos KR_1 - \chi \sin KR_1)(\chi + ik)} \xrightarrow{\chi(R_2 - R_1)} \\ &\xrightarrow{\chi(R_2 - R_1)} e^{-2ikR_2} \frac{\chi + ik}{\chi - ik} \end{aligned} \quad (7)$$

(one can easily see that  $|S| = 1$ ) and

$$\begin{aligned} A &= e^{-2ikR_2} \frac{2ik\chi e^{2\chi(R_1 - R_2)}}{(K \cos KR_1 + \chi \sin KR_1)(\chi - ik) + e^{2\chi(R_1 - R_2)}(K \cos KR_1 - \chi \sin KR_1)(\chi + ik)} \xrightarrow{\chi(R_2 - R_1)} \\ &\xrightarrow{\chi(R_2 - R_1)} e^{-ikR_2} \frac{2ik\chi e^{\chi(R_1 - R_2)}}{[K \cos KR_1 + \chi \sin KR_1](\chi - ik)} \rightarrow 0 \end{aligned} \quad (8)$$

### 3. Subdivision of scattering into a sequence of two successive processes

We will describe the impact of the particles with this potential as a sequence of two successive processes: in the first stage we think to an ingoing wave packet impinging from outside on the barrier, producing a reflected wave in the external region (I), tunneling through the barrier, and finally penetrating in the well where it is represented by an ingoing mode. In the second phase, we will consider the presence of an outgoing wave from the well (III), which, after a reflection against the internal side of the barrier, tunnels through the barrier and produces, finally, an outgoing mode in the external region (I).

The scheme of these processes is sketched in Fig. 2.

where  $j_l(kr)$  is the spherical Bessel function. For the potential depicted in Fig.1 the stationary function  $\psi_l(k, r)$  is equal to

$$\begin{aligned} \psi_{I,l}(k, r) &= (1/2)[h_l^{(2)}(kr) + S_l h_l^{(1)}(kr)] \\ \psi_{II,l}(k, r) &= (1/2)[- \alpha_l h_l^{(1)}(i\chi r) + \beta_l h_l^{(2)}(i\chi r)] \end{aligned} \quad (4)$$

$$\psi_{III,l}(k, r) = (1/2) A_l j_l(Kr)$$

in the regions I, II and III, respectively. Here  $h_l^{(1,2)}(kr)$  are the spherical Hankel functions of the first and second kind, respectively,

$$\chi = [2m(V_1 - E)]^{(1/2)} / \hbar, \quad K = [2m(V_0 + E)]^{(1/2)} / \hbar.$$

The coefficients  $S_l$  (the  $S$ -matrix),  $\alpha_l$ ,  $\beta_l$  and  $A_l$  can be easily found explicitly analytically utilizing the continuity relations of  $\psi_l(k, r)$  and  $d\psi_l(k, r)/dr$  at points  $R_1$  and  $R_2$ .

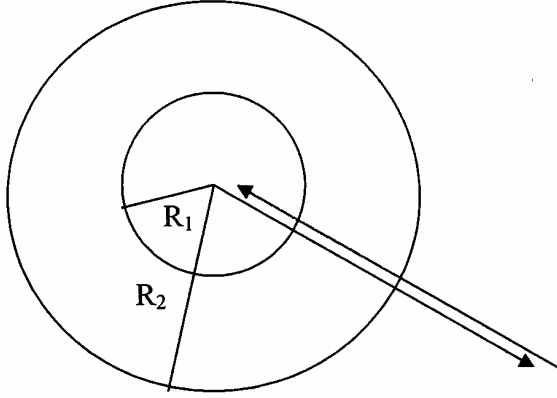


Fig. 2. Schematic description of the impact process.

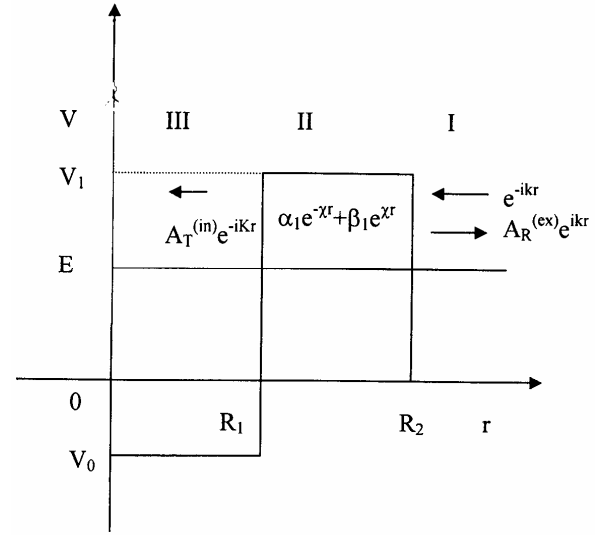


Fig. 3. Schematic description of the impact of a wave packet with a barrier from outside.

### i) Impact from outside

We will start by considering an initial wave packet, defined in the outer region (I) by means of a superposition of ingoing spherical  $s$ -waves, and moving from outside toward the barrier region (II) where the potential has value  $V_1$  (a schematic stationary description of the impact from outside see in Fig. 3):

$$\Phi_0^{(1)}(r, t) = \int_0^{V_1} dE g(E) e^{-ikr - iEt/\hbar} \quad (R_2 \leq r < \infty). \quad (9)$$

When the wave encounters the barrier, it is partially reflected with wave packet:

$$\Phi_R^{(1)}(r, t) = \int_0^{V_1} dE g(E) A_R^{(ex)} e^{-ikr - iEt/\hbar} \quad (R_2 \leq r < \infty). \quad (10)$$

At the same time, the wave packet tunnels through the barrier in region (II), where it is represented by the function:

$$\Phi_{II}^{(1)}(r, t) = \int_0^{V_1} dE g(E) (\alpha_1 e^{-zr} + \beta_1 e^{zr}) e^{-iEt/\hbar} \quad (R_1 \leq r < R_2). \quad (11)$$

After the tunnelling through the barrier, the wave packet penetrates inside the well and in region (III) we can write

$$\Phi_T^{(1)}(r, t) = \int_0^{V_1} dE g(E) A_T^{(in)} e^{-iKr - iEt/\hbar} \quad (0 < r < R_1). \quad (12)$$

In general we can say that the wave packet is described by the expression

$$\Phi_J^{(1)}(r, t) = \int_0^{V_1} dE g(E) \phi_J^{(1)}(k, r) e^{-iEt/\hbar}, \quad (13)$$

where the index  $J$  is  $I, II, III, 0, T$  or  $R$  depending on the particular mode considered,  $\phi_J^{(1)}$  being the stationary wave functions:

$$\begin{aligned} \phi_I^{(1)} &= e^{-ikr} + A_R^{(ex)} e^{ikr} & R_2 \leq r < \infty, \\ \phi_{II}^{(1)} &= \alpha_1 e^{-zr} + \beta_1 e^{zr} & R_1 \leq r \leq R_2, \\ \phi_{III}^{(1)} &= A_T^{(in)} e^{-iKr} & 0 < r \leq R_2. \end{aligned} \quad (14)$$

Furthermore,  $g(E)$  is a normalized amplitude weight factor such as:

$$\int_0^{V_1} dE |g(E)|^2 = N, \quad N < \infty. \quad (15)$$

We normalize equation (9) by the condition

$$4\pi \int_{R_2}^{\infty} dr |\Phi_0^{(1)}(r,t)|^2 = 1 \quad (16)$$

which is strictly possible only in the limit  $V_1 \rightarrow \infty$ . For the sake of simplicity we neglect the contribution of the energy above the barrier, considering only cases where  $E < V_1$ .

In addition, we have that  $A_R^{(ex)}$ ,  $\alpha_I$ ,  $\beta_I$  and  $A_T^{(in)}$  are, respectively, the external reflection amplitude factor, the evanescent and anti-evanescent wave amplitude factors during the first tunnelling and the internal transmission amplitude factor. Their analytical expression can be found by imposing the continuity condition for both the stationary wave functions and their first derivatives at the points  $r = R_2$  and  $r = R_1$ , finding for the external reflection coefficient the expression

$$A_R^{ex} = -e^{-2ikR_2} \frac{(\chi + ik)(iK + \chi) + e^{-2\chi(R_2-R_1)}(\chi - ik)(iK - \chi)}{(\chi - ik)(iK - \chi) + e^{-2\chi(R_2-R_1)}(\chi + ik)(iK + \chi)} \quad (17)$$

which, in the limit  $\chi(R_2 - R_1) \rightarrow \infty$ , becomes  $-e^{-2ikR_2} \frac{\chi + ik}{\chi - ik}$ .

For the internal transmission coefficient we get

$$A_T^{in} = e^{-ikR_2 + iKR_1} \frac{4ik\chi e^{-\chi(R_2-R_1)}}{(\chi - ik)(iK - \chi) + e^{-2\chi(R_2-R_1)}(\chi + ik)(iK + \chi)} \quad (18)$$

that, in the same limit as before, tends to 0.

The calculation of the probability fluxes can be used for controlling the conservation law.

The fluxes  $j_I^{(1)}$ ,  $j_{II}^{(1)}$  and  $j_{III}^{(1)}$  in the three regions *I*, *II* and *III* are in general defined as

$$j_n^{(1)} = \frac{i\hbar}{2m} (\Phi_n^{(1)} \frac{d}{dr} (\Phi_n^{(1)*}) - \Phi_n^{(1)*} \frac{d}{dr} (\Phi_n^{(1)})), \quad n = I, II, III. \quad (19)$$

One can use the approximation

$$\int_0^{\infty} j_n^{(1)}(r,t) dt \approx \int_{-\infty}^{\infty} j_n^{(1)}(r,t) dt. \quad (20)$$

For quasi-monochromatic wave packet centred around the value  $E = \langle E \rangle$ , i.e. when  $|g(E)|^2$  is a delta function  $\delta(E - \langle E \rangle)$ , with  $\langle E \rangle$  in the open interval  $(0, V_1)$ , one obtains

$$\frac{\int_{-\infty}^{\infty} j_R^{(1)}(r,t) dt}{\int_{-\infty}^{\infty} j_0^{(1)}(r,t) dt} \approx \left| A_R^{(ex)}(\langle E \rangle) \right|^2, \quad (21)$$

$$\frac{\int_{-\infty}^{\infty} j_{II}^{(1)}(r,t) dt}{\int_{-\infty}^{\infty} j_0^{(1)}(r,t) dt} \approx \left| \alpha_I(\langle E \rangle) e^{-\chi(\langle E \rangle)r} + \beta_I(\langle E \rangle) e^{\chi(\langle E \rangle)r} \right|^2,$$

$$\frac{\int_{-\infty}^{\infty} j_{III}^{(1)}(r,t)dt}{\int_{-\infty}^{\infty} j_0^{(1)}(r,t)dt} \approx \frac{K(\langle E \rangle)}{k(\langle E \rangle)} \left| A_T^{(in)}(\langle E \rangle) \right|^2.$$

Hence, from the conservation law for the probability fluxes

$$\int_{-\infty}^{\infty} j_0^{(1)}(r,t)dt = \int_{-\infty}^{\infty} j_R^{(1)}(r,t)dt + \int_{-\infty}^{\infty} j_T^{(1)}(r,t)dt \quad (22)$$

one obtains:

$$\left| A_R^{(ex)}(\langle E \rangle) \right|^2 + \frac{\kappa(\langle E \rangle)}{k(\langle E \rangle)} \left| A_T^{(in)}(\langle E \rangle) \right|^2 = 1 \quad (23)$$

that for  $V_0 = 0$  becomes

$$\left| A_R^{(ex)}(\langle E \rangle) \right|^2 + \left| A_T^{(in)}(\langle E \rangle) \right|^2 = 1. \quad (24)$$

The phase times  $\tau_T^{ph(in)}$  and  $\tau_R^{ph(ex)}$  can be defined as the evident generalization of the one-dimensional (1-D) definitions

$$\begin{aligned} \tau_T^{ph(in)} &= \hbar \frac{\partial(\arg A_T^{(in)}(E) e^{-iKR_1})}{\partial E} - \frac{-R_1}{v} = \hbar \frac{\partial(\arg(A_T^{(in)}(E)))}{\partial E} \\ &= \hbar \frac{\partial \arg A_T^{(in)}(E)}{\partial E} \xrightarrow{\chi(R_2-R_1)} \left( \frac{1}{v} + \frac{1}{V} \right) / \chi, \quad V = \frac{\hbar K}{m} \end{aligned} \quad (25)$$

and

$$\tau_R^{ph(ex)} = \hbar \frac{\partial(\arg(A_R^{(ex)}(E) e^{-iKR_2}))}{\partial E} - \frac{-R_2}{v} = \hbar \frac{\partial(\arg(A_R^{(ex)}(E)))}{\partial E} \xrightarrow{\chi(R_2-R_1)} \frac{2}{v\chi}, \quad (26)$$

where  $v = \hbar k / m$ .

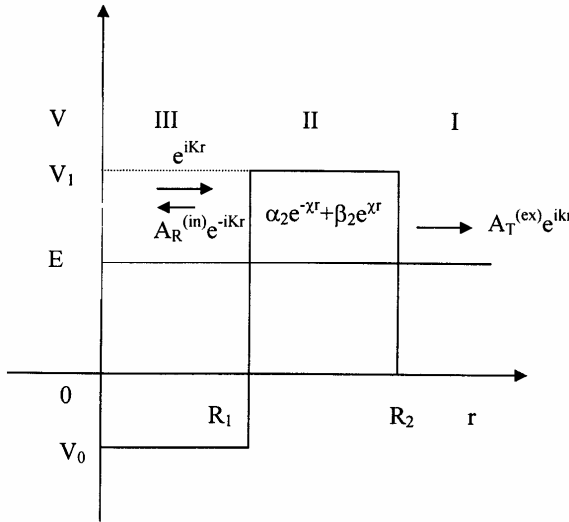


Fig. 4. Schematic view of the second phase of the scattering process – the emission from inside.

So we can see from the previous expressions, for analogy with the same one-dimensional quantities, the manifestation of the Hartmann effect and also the absence of dependence on the geometrical characteristics of the barrier  $R_1$  and  $R_2$ .

*ii) Emission out from the barrier*

Now we shall study the evolution of a wave coming out from the central core of the system, with a wave function given by

$$\Phi^{(in)}(r,t) = \int_0^{V_1} dE \tilde{g}(E) e^{ikr - iEt/\hbar} \quad (0 \leq r \leq R_1) \quad (27)$$

constructed overlapping stationary solutions propagating in the positive  $r$ -direction from the well region. In Fig. 4 the scheme of the various waves is presented.

When the wave impinges the barrier from inside, a reflected wave is formed, whose wave function

$$\Phi_R^{(2)}(r,t) = \int_0^{V_1} dE \tilde{g}(E) A_R^{(in)} e^{-iKr - iEt/\hbar} \quad (0 \leq r \leq R_1). \quad (28)$$

Afterwards, in the region (II) of the barrier a system of evanescent and anti-evanescent waves

$$\Phi_{II}^{(2)}(r, t) = \int_0^{V_1} dE \tilde{g}(E) (\alpha_2 e^{-\chi r} + \beta_2 e^{\chi r}) e^{-iEt/\hbar} \quad (R_1 \leq r \leq R_2) \quad (29)$$

while in the outer region a propagating wave will exist

$$\Phi_I^{(2)}(r, t) = \int_0^{V_1} dE \tilde{g}(E) A_T^{(ex)} e^{ikr - iEt/\hbar} \quad (R_2 \leq r < \infty) . \quad (30)$$

As in the previous case we have that  $\tilde{g}(E)$  is a normalized amplitude weight factor, and  $A_R^{(in)}$ ,  $\alpha_2$ ,  $\beta_2$  and  $A_T^{(ex)}$  are the internal reflection, the evanescent and anti-evanescent and the external transmission amplitude factors, respectively.

With calculations similar to that of the first part we obtain the explicit expressions of the amplitude factors

$$A_R^{in} = -e^{-2iKR_1} \frac{(\chi + iK)(ik - \chi) + e^{2\chi(R_1 - R_2)}(\chi - iK)(ik + \chi)}{(\chi - iK)(ik - \chi) + e^{2\chi(R_1 - R_2)}(\chi + iK)(ik + \chi)} \quad (31)$$

and

$$A_T^{ex} = e^{-ikR_2 + iKR_1} \frac{4iK\chi e^{-\chi(R_2 - R_1)}}{(\chi - iK)(ik - \chi) + e^{-2\chi(R_2 - R_1)}(\chi + iK)(ik + \chi)} . \quad (32)$$

By the way, it is easily seen that  $A_T^{(in)} = \frac{k}{K} A_T^{(ex)}$ .

Also in this case we can demonstrate the conservation of the current fluxes

$$\left| A_R^{(in)}(\langle E \rangle) \right|^2 + \frac{k(\langle E \rangle)}{K(\langle E \rangle)} \left| A_T^{(ex)}(\langle E \rangle) \right|^2 = 1 . \quad (33)$$

The phase times  $\tau_R^{Ph(in)}$  and  $\tau_T^{Ph(ex)}$  can be also defined by the evident generalization of the general 1-D definition from [1, 2]

$$\tau_R^{Ph(in)} = \hbar \frac{\partial(\arg(A_R^{(in)}(E)e^{-ikR_2}))}{\partial E} - \frac{R_1}{v} \xrightarrow{\chi(R_2 - R_1)} \frac{2}{\chi V} \quad (34)$$

and

$$\tau_T^{Ph(ex)} = \hbar \frac{\partial(\arg(A_T^{(ex)}(E)e^{-ikR_2}))}{\partial E} - \frac{R_1}{v} = \hbar \frac{\partial(\arg(A_T^{(ex)}(E)))}{\partial E} \equiv \tau_T^{Ph(in)} \xrightarrow{\chi(R_2 - R_1)} \left(\frac{1}{v} + \frac{1}{V}\right) / \chi . \quad (35)$$

So, we see from (34) - (35) again the manifestation of the Hartman effect with the absence of the dependence on  $R_1$  and  $R_2$ .

*iii) Connection between both schemes.*

Finally we shall connect the two mechanisms of scattering described above in one single scattering event, introducing the  $S$ -matrix of scattering and considering the multiple reflections inside the potential well and utilizing the formulas (6) - (8). Comparing them with the expressions (17) and (18), (31) and (32), one can easily see that

$$A = \frac{A_T^{(in)}}{1 + A_R^{in}} \quad (36)$$

and

$$S = -A_R^{(ex)} - iAA_T^{(ex)} = -A_R^{(ex)} + \frac{A_T^{(ex)} A_T^{(in)}}{1 + A_R^{(in)}} . \quad (37)$$

The physical meaning of the term  $\frac{1}{1 + A_R^{in}}$  is directly connected to the presence of a infinite sequence of multiple internal reflections that can be described by the stationary wave functions

$$A_T^{(in)} (1 - A_R^{(in)} + (A_R^{(in)})^2 - (A_R^{(in)})^3 + \dots) e^{-ikr} = \frac{A_T^{(in)}}{1 + A_T^{(in)}} e^{-ikr}, \quad (38)$$

$$A_T^{(in)} (1 - A_R^{(in)} + (A_R^{(in)})^2 - (A_R^{(in)})^3 + \dots) e^{ikr} = \frac{A_T^{(in)}}{1 + A_T^{(in)}} e^{ikr} \quad (39)$$

for the ingoing and the outgoing waves respectively.

The scattering phase time is

$$\tau_{sc}^{ph} = \hbar \frac{\partial \arg S e^{ikR_2}}{\partial E} - \left(-\frac{R_2}{v}\right) \quad (40)$$

and the limit  $\chi(R_2 - R_1)$  approaching infinity it goes to  $2/(v\chi)$ . So in this limit, the scattering phase time coincides with  $\tau_R^{ph(ex)}$ .

*iv) Resonances*

In all the previous analysis we have disregarded the possibility that resonances develop during tunneling and scattering.

To take into account the insurgence of resonances we rewrite the scattering matrix S in the following form

$$S = \alpha \frac{A_1 \chi + ikA_2}{A_1 \chi - ikA_2} \quad (41)$$

with, from the comparison with (7) we set:  $\alpha = e^{-2ikR_2}$ ,  $A_1 = B_1 + e^{-2\chi(R_2 - R_1)} B_2$ ,  $A_2 = B_1 - e^{-2\chi(R_2 - R_1)} B_2$ ,  $B_1 = K \cos(KR_1) + \chi \sin(KR_1)$ ,  $B_2 = K \cos(KR_1) - \chi \sin(KR_1)$ .

In the region of a resonance, we can develop S into a series of powers of  $(E - E_r)$ ,  $E_r$  being the eigenvalue of the energy for the resonance considered, solution of the transcendental equation  $A_1(E_r) = 0$ , obtaining

$$S = \alpha \frac{\left(\frac{\partial(A_1 \chi)}{\partial E}\right)_{E=E_r} (E - E_r) + i(kA_2)_{E=E_r}}{\left(\frac{\partial(A_1 \chi)}{\partial E}\right)_{E=E_r} (E - E_r) - i(kA_2)_{E=E_r}}, \quad (42)$$

that shows better its resonant character after rewriting it in the form

$$S = \alpha \frac{E - E_r - i\Gamma/2}{E - E_r + i\Gamma/2} \quad (43)$$

with

$$\frac{\Gamma}{2} = \frac{-k(E_r)A_2(E_r)}{\chi(E_r)\left(\frac{\partial A_1}{\partial E}\right)_{E=E_r}} = \frac{2k(E_r)e^{-2\chi(E_r)(R_2 - R_1)} B_2(E_r)}{\chi(E_r)\left(\frac{\partial A_1}{\partial E}\right)_{E=E_r}}. \quad (44)$$

If  $\chi(R_2 - R_1)$  is very large, we can neglect all the terms containing the negative exponential factor (with the only exception in the numerator in (44)) and write finally

$$\frac{\Gamma}{2} = 2k(E_r)e^{-2\chi(E_r)(R_2 - R_1)} F(E_r) \quad (45)$$

with

$$F(E_r) = \frac{\hbar K(E_r)}{\chi(E_r)\sqrt{m/2}\left(\frac{1}{\sqrt{V_0 + E_r}}(1 + \chi(E_r)R_1) + \frac{1}{\sqrt{V_1 - E_r}}(1 + K(E_r)R_1)\right)}. \quad (46)$$

As it is known, a narrow resonance is connected with the Breit - Wigner expressions in the cross section and in the time delay for quasi-monochromatic particles and with an exponential decay of the compound system in time at any fixed position  $r > R_2$ , with a time constant  $\tau = \hbar / \Gamma$ .

#### 4. Perspectives and conclusions

We have derived and analyzed the general expressions for the elastic scattering S-matrix, for all transmission and reflection probability amplitudes and the connection between them in a 3-D tunneling process through a spherically symmetric barrier taking into account the multiple internal reflection.

We have also written the tunneling and reflection phase time, showing, on the basis of their expression, the occurrence of the Hartmann effect.

We have also taken into account the resonances and written in this case the S- matrix, with the resonance width.

The present analysis is an initial step for the comprehensive phenomenological time-dependent study of the emission of protons or alpha-particles from a spherical compound nucleus. In these cases we can initiate from a simple coulombian barrier

$$V_c = Z_1 Z_2 e^2 / r$$

( $Z_1 e$  and  $Z_2 e$  are the charges of a daughter nucleus and an emitted particle, respectively) instead of a rectangular potential barrier. And instead of the functions  $h_l^{(1,2)}(kr)$  we have to use the coulombian functions  $G_l(k, \eta, r) \pm i F_l(k, \eta, r)$  in the field of the coulombian barrier with

$$F_0(k, \eta, r) \rightarrow (kr)^{-1} \sin(kr - \eta \ln 2kr + \sigma),$$

$$r \rightarrow \infty$$

$$G_0(k, \eta, r) \rightarrow (kr)^{-1} \cos(kr - \eta \ln 2kr + \sigma)$$

$$r \rightarrow \infty$$

(where  $\eta = Z_1 Z_2 e^2 m / \hbar^2 k$ ,  $\sigma = \arg \Gamma(1 + i\eta)$ ). Of course, in the realistic situations one has to use typical phenomenological potentials (like the Saxon - Woods well) and consider also the charge distribution inside and at the surface of a daughter nucleus.

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### ТУННЕЛИРОВАНИЕ И РАССЕЯНИЕ ЧАСТИЦ В СФЕРИЧЕСКОМ ТРЕХМЕРНОМ ПОТЕНЦИАЛЕ С БАРЬЕРОМ

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Изучены туннелирование и рассеяние частиц в трёхмерном прямоугольном барьере, содержащем внутри потенциальную прямоугольную яму. Учтены явно многократные внутренние отражения внутри потенциальной ямы. Получены и исследованы точные аналитические соотношения между *S-матрицей* упругого рассеяния и всеми амплитудами вероятности (*внешнего и внутреннего отражений от барьера, туннелирования внутрь и туннелирования наружу*). Изучены соответствующие фазовые времена. Обсуждены конкретные перспективы обобщения на случай кулоновских барьеров и непрямоугольных (размытых) потенциальных ям.

## ТУНЕЛЮВАННЯ ТА РОЗСІЯННЯ ЧАСТИНОК У СФЕРИЧНОМУ ТРИВИМІРНОМУ ПОТЕНЦІАЛІ З БАР'ЄРОМ

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Вивчено тунелювання та розсіяння частинок у тривимірному прямокутному бар'єрі, що містить усередині прямокутну потенціальну яму. Враховано явним чином багаторазові внутрішні відбиття всередині потенціальної ями. Одержано та досліджено точні аналітичні співвідношення між  $S$ -матрицею пружного розсіяння та всіма амплітудами ймовірності (зовнішнього та внутрішнього відбиття від бар'єра, тунелювання всередину та зовні). Вивчено відповідні фазові часи. Обговорено конкретні перспективи узагальнення на випадок кулонівських бар'єрів та непрямокутних (розмитих) потенціальних ям.