Single- and double-phonon giant monopole resonances in a nonlinear approach

V. Yu. Denisov[1,2,*] and S. Yamaji[1,†]
[1]*Institute of Physical and Chemical Research (RIKEN), 2-1 Hirosawa, Wako, Saitama 351-0198, Japan*
[2]*Institute for Nuclear Research, Prospect Nauki 47, 252028 Kiev, Ukraine*[‡]
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Isoscalar monopole density vibrations in spherical nuclei with a sharp surface are studied in a nonlinear hydrodynamical approach. The frequency shift of the one- and two-phonon excitations due to nonlinear terms is obtained. The frequencies of one- (two-) phonon giant isoscalar monopole resonances calculated in the nonlinear hydrodynamic theory increase by $\approx 2\%(8\%)$ in heavy nuclei and by ≈ 14–18% (50%) in light nuclei compared to the linear approximation. The frequency shift is a function of both the mass number A and the parameters of the nucleon-nucleon interaction.

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I. INTRODUCTION

Giant dipole resonances were predicted [1] and experimentally discovered [2] more than 50 years ago. Nevertheless, the giant resonances have been and still are a major topic of research in nuclear physics [1–29]. Many different types of giant resonances have been discovered [3–6]. The isoscalar giant monopole resonance (GMR) is of particular interest because its energy is directly related to the compressibility of nuclear matter [3–11]. The compressibility of nuclear matter is one of the most important quantities, because it influences both the ground and excited state properties of nuclei, heavy-ion reactions, properties of neutron stars, and supernova explosions [30,31].

In order to extract the compressibility value it is necessary to know the experimental energies of the GMR over a wide range of atomic mass number A [7–10]. From a macroscopic analysis of the A dependence of the GMR energy it is possible to extract the compressibility of nuclear matter [9]. The extracted value of the compressibility depends on the model applied in the analysis of the experimental data. Nonlinear effects may give contributions to the energy of the GMR [7,15,17,18]. Therefore nonlinear effects may change the value of the compressibility obtained from an analysis of experimental GMR energies.

In the past few years two-phonon giant resonances have also been studied both experimentally and theoretically [18–29] The two-phonon giant dipole resonance has been observed at an energy about twice as large as that of the one-phonon resonance [19–24].

The one-phonon giant resonance is treated as an excited state built on the ground state of the nucleus. In the framework of the harmonic oscillator model the two-phonon giant resonance is treated as an excited state built on a one-phonon excitation. Decay schemes of the one- and two-phonon giant resonances confirm this origin of these resonances [19–24]. However, the measured energies of the two-phonon isovector giant dipole resonances are slightly smaller than twice the one-phonon resonance energy [19,20]. Such a reduction in the excitation energy of the two-phonon state may be caused by the anharmonicity of vibrations in nuclear matter.

The purpose of this paper is to study the frequency shift of the one- and two-phonon isoscalar GMR due to the anharmonicity of density oscillations in a spherical nucleus with a sharp surface in the framework of the nonlinear hydrodynamical model. We will obtain an analytical expression for the frequency shift of the one- and two-phonon GMR caused by nonlinearities. Therefore we consider giant resonances in a spherical nucleus with a sharp surface. It should be noted that various properties of giant resonances have been successfully described by the use of different linear hydrodynamical approximations [3,4,6,7,11,12,14]. The anharmonicities of vibrations in a nuclear matter slab with sharp surfaces has been considered within a nonlinear hydrodynamical model in Ref. [13].

The nonlinearity of the one-phonon isoscalar GMR was investigated in the quantized time-dependent Hartree-Fock (QTDHF) model [15], in a random phase approximation (RPA) based on the Hartree-Fock-Bogoliubov (HFB+RPA) nuclear basis states [7], in a relativistic mean-field (RMF) theory [17], and in an extended RPA (ERPA) model [18]. The anharmonicity of the two-phonon giant resonances was estimated in the RPA models in [16,18,22]. These studies require cumbersome numerical calculations and the link between the parameters of the nucleon-nucleon interaction and the shift of resonance energy caused by anharmonicity can only be obtained numerically. However, in [26] the anharmonicity of the two-phonon giant resonances was studied analytically in a variational approach with a simplified interaction.

The analytical expression derived in this paper confirms the direct and transparent connection between the shift in energy of the one- and two-phonon giant resonances due to anharmonicities. Also illustrated is the dependence of these shifts on values of the constants of the Skyrme-type [4,32,33] energy density functional. The constants of the energy density functional are related to the parameters of the nucleon-nucleon interaction [4,32–37].

The nonlinear hydrodynamic model of density vibrations in spherical nuclei is described in Sec. II. The results for the

*Electronic address: denisov@kinr.kiev.ua
†Electronic address: yamajis@rikaxp.riken.go.jp
‡Permanent address.

isoscalar GMR are presented in Sec. III. The summary and conclusions are presented in Sec. IV.

II. NONLINEAR DENSITY VIBRATIONS IN A SPHERICAL NUCLEUS

Density oscillations $\rho(r,t)$ in the hydrodynamic approximation obey the continuity equation [4,38]

$$\frac{\partial \rho}{\partial t} + \mathrm{div}(\rho \mathbf{v}) = 0 \tag{1}$$

and the Euler equation [4,38]

$$\frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v}\nabla)\mathbf{v} + \frac{1}{m}\boldsymbol{\nabla}\frac{\delta \mathcal{E}(\rho)}{\delta \rho} = 0, \tag{2}$$

where $\mathbf{v}(r,t)$ is the velocity of nucleons, $\delta\mathcal{E}(\rho)/\delta\rho$ is the variational derivative of the energy density functional, and m is the nucleon mass.

Let us consider the following energy density functional of nuclear matter:

$$\mathcal{E}(\rho) = \rho\left(\mathcal{E}_\infty + a_0 \frac{(\rho-\rho_\infty)^2}{2\rho_\infty^2} + b_0 \frac{(\rho-\rho_\infty)^3}{6\rho_\infty^3} + c_0 \frac{(\rho-\rho_\infty)^4}{12\rho_\infty^4} \right) + d_0 \frac{(\nabla\rho)^2}{2\rho_\infty}, \tag{3}$$

where $a_0 = K/9$, b_0, c_0, and d_0 are constants, and K is the nuclear matter compression modulus [2–7]. We should note that the form of realistic energy density functionals [32,33] does not coincide exactly with Eq. (3). However, any realistic energy density functional per nucleon, $\mathcal{E}(\rho)/\rho$, may be expanded into a power series in the deviation of the density from the equilibrium value ρ_∞ as in Eq. (3), when there are no gradient terms. The form of the gradient term in Eq. (3) is the same as the gradient term of the energy density functional of the Skyrme force in the Thomas-Fermi approach [33].

In the case of monopole vibrations in spherical nuclei, Eqs. (1) and (2) may be simplified,

$$\frac{\partial \xi}{\partial t} + \frac{1}{r^2}\frac{\partial (r^2 v)}{\partial r} + \frac{f}{r^2}\frac{\partial (r^2 \xi v)}{\partial r} = 0, \tag{4}$$

$$\frac{\partial v}{\partial t} + \frac{a}{m}\frac{\partial \xi}{\partial r} - \frac{d}{m}\frac{\partial}{\partial r}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \xi}{\partial r} \right) \right] + ev\frac{\partial v}{\partial r} + \frac{b}{m}\xi\frac{\partial \xi}{\partial r} + \frac{c}{m}\xi^2\frac{\partial \xi}{\partial r} = 0, \tag{5}$$

where e and f are the auxiliary constants ($e=f=1$), $a = a_0$, $b = 3a_0 + b_0$, $c = 2b_0 + c_0$, $d = d_0$, and ξ is the function describing the density vibrations:

$$\rho(r,t) = \rho_\infty[1 + \xi(r,t)]. \tag{6}$$

We neglect fourth and higher order terms in Eq. (5).

Equations (4) and (5) are nonlinear. The nonlinear terms which contain the constants e and f in Eqs. (4) and (5) arise due to the nonlinear nature of the hydrodynamic equations. The terms with constants b and c in Eqs. (4) and (5) are related to the anharmonic potential nature of the energy density functional (3).

Let us obtain the solution to the system of Eqs. (4) and (5) in the form of a perturbation series, a treatment which is similar to the treatment of nonlinear oscillations in classical mechanics [39]:

$$\xi = \xi_0 + \xi_1 + \xi_2 + \cdots, \tag{7}$$

$$v = v_0 + v_1 + v_2 + \cdots, \tag{8}$$

$$k = k_0 + k_1 + k_2 + \cdots, \tag{9}$$

$$\omega = \omega_0 + \omega_1 + \omega_2 + \cdots. \tag{10}$$

As a first approximation the system of equations (4) and (5) can be easily transformed by an equation for density vibrations:

$$\frac{\partial^2 \xi_0}{\partial t^2} - \frac{a}{m}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \xi_0}{\partial r} \right) + \frac{d}{m}\frac{1}{r^2}\frac{\partial}{\partial r}\left\{ r^2 \frac{\partial}{\partial r}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 \frac{\partial \xi_0}{\partial r} \right) \right] \right\} = 0. \tag{11}$$

Let us consider the link between our hydrodynamic approach and microscopic theories of giant resonances. Microscopic theories of giant resonances are based on the RPA [4]. The Landau-Vlasov equation is the semiclassical analog of the microscopic RPA equations [4]. The nuclear fluid dynamics model [4,14] can be derived by using the Landau-Vlasov equation. The density fluctuations in the nuclear fluid dynamics model also obey the continuity equation (1). The equations for monopole density fluctuations in a nuclear fluid dynamics model in the case of quadrupole distortions of the local Fermi sphere [14] are similar to Eqs. (4), (5), and (11). Note that the monopole density oscillations are purely longitudinal. So the terms related to transversal motion are absent in the fluid dynamic equations in this case. The difference between Eq. (11) and the fluid dynamics density vibration equation is connected to the value of the constants a and d in Eq. (11) (see also [14]). Therefore, a hydrodynamic model of the GMR is an approximation to a microscopic theory based on the RPA. So it is reasonable to examine anharmonic effects in the GMR by studying the density fluctuations in the framework of a nonlinear hydrodynamics model.

The solution to Eqs. (4) and (5) in first order perturbation theory is

$$\xi_0 = \alpha \sin(\omega_0 t)\frac{\sin(k_0 r)}{k_0 r} = \alpha(t)\frac{\sin(k_0 r)}{k_0 r}, \tag{12}$$

$$v_0 = -\alpha\frac{\omega_0}{k_0}\cos(\omega_0 t)\frac{\sin(k_0 r) - k_0 r\cos(k_0 r)}{(k_0 r)^2}. \tag{13}$$

Here α is the amplitude, and ω_0 and k_0 are the frequency and wave number of oscillations, respectively. The solutions obtained in first order perturbation theory are linear in α. The frequency and wave number are connected by the dispersion equation

$$\omega_0^2=\frac{a}{m}k_0^2+\frac{d}{m}k_0^4. \tag{14}$$

Note that we chose the solution (12) and (13) of Eqs. (4) and (5), because this solution and dispersion relation (14) are also valid in the limit $d\to 0$, which limit is often considered for giant monopole density resonances (see, for example, [3,4]).

The density oscillation wave number k_0 is determinated by kinematic and dynamic linear boundary conditions on the free surface of the nucleus.

The kinematic boundary condition is assumed to be the equality of the normal component of the nucleon velocity on the surface, $v(R,t)$, to the surface velocity $v_S(t)$,

$$v(R,t)=v_S(t), \tag{15}$$

where $v_S(t)=\partial\delta(t)/\partial t$, R is the nuclear radius, and $\delta(t)$ is the amplitude of surface oscillations. In the case of a spherical nucleus $v(R,t)$ coincides with the radial velocity in the linear approximation.

The dynamical boundary condition is the equality of the normal component of the surface effective pressure $\sigma(r,t)$, caused by the variation of the nucleon density $\xi(r,t)$ in a nucleus, to the pressure $P(t)$ caused by both the surface tension and the shift of the surface $\delta(t)$ from the equilibrium position,

$$\sigma(R,t)=P(t). \tag{16}$$

We neglect the surface tension $P(t)=0$, because the influence of the surface restoring force on the GMR energy is small [3,7,11]. Note that boundary conditions (15) and (16) are the same as used in Secs. 6A-3a and 6A-3b in [3] and in [7,11,12].

Let us substitute solutions (12) and (13) into the boundary conditions (15) and (16) and obtain the equation for the wave number k_0:

$$\begin{aligned}\sigma(r,t)\big|_{r=R}&=\rho_\infty\frac{\delta\mathcal{E}(\rho)}{\delta\rho}\bigg|_{r=R}\\&=m\rho_\infty\left(a\,\xi(r,t)-d\,\frac{\partial^2\xi(r,t)}{\partial r^2}\right)\bigg|_{r=R}\\&=\alpha\rho_\infty m(a+dk_0^2)\sin(\omega_0 t)\,\sin(k_0 r)\big|_{r=R}\\&\propto\sin(k_0R)=0.\end{aligned} \tag{17}$$

As a result of to the dispersion relation (14), the frequency of vibrations ω_0 is also fixed by the boundary conditions. The amplitude of vibrations is obtained from the condition of energy equality, evaluated by means of the quantum and classical expressions,

$$\hbar\omega_0=E_{class}. \tag{18}$$

The vibration energy calculated in the classical approach is

$$\begin{aligned}E_{class}&=\frac{1}{2}\int dVm\rho_\infty v_0^2+\frac{1}{2}\int dV\rho_\infty[a\xi_0^2+d(\nabla\xi_0)^2]\\&=\frac{1}{2}\mathcal{B}[\dot{\alpha(t)}]^2+\frac{1}{2}\mathcal{C}[\alpha(t)]^2.\end{aligned} \tag{19}$$

By taking into account Eqs. (12) and (13), the expression for harmonic oscillator frequency $\omega_0=(\mathcal{C}/\mathcal{B})^{1/2}$, and boundary condition (17) we find the vibration amplitude

$$\begin{aligned}\alpha&=\left(\frac{2\hbar}{(\mathcal{BC})^{1/2}}\right)^{1/2}\\&=\pi\left(\frac{2\pi\hbar}{3[mr_0^2(a+\pi^2dr_0^{-2}A^{-2/3})]^{1/2}}\right)^{1/2}A^{-2/3},\end{aligned} \tag{20}$$

where A is the number of the nucleons in the nucleus and we use $R=r_0A^{1/3}$.

Let us consider the next order of the perturbation approximation.

The nuclear interaction within the nucleus volume is much stronger than that on its surfaces. Therefore we may consider that the nonlinear density vibration dynamics in the nucleus is determined by the dynamics within the volume, and that the oscillations of the surface are controlled by the volume dynamics. We shall ignore the nonlinearities in the boundary conditions. One can see that we should take into account the $\hbar^2$ and $\hbar^4$ corrections to the Thomas-Fermi kinetic energy in order to describe the surface layer of nuclear matter accurately [33]. We have restricted ourselves to the Thomas-Fermi approach to the kinetic energy functional in Eq. (3). Therefore, the wave number k is fixed in the next orders of the perturbation theory to be the same as in the linear approximation, i.e., $k=k_0$.

The terms with coefficients b, e, and f in Eqs. (4) and (5) should be taken into account in the second order of perturbation analysis. The system of equations determining ξ_1 and v_1 is

$$\frac{\partial\xi_1}{\partial t}+\frac{1}{r^2}\frac{\partial(r^2v_1)}{\partial r}=-\frac{f}{r^2}\frac{\partial(r^2\xi_0v_0)}{\partial r}, \tag{21}$$

$$\frac{\partial v_1}{\partial t}+\frac{a}{m}\frac{\partial\xi_1}{\partial r}-\frac{d}{m}\frac{\partial}{\partial r}\left(\frac{1}{r^2}\frac{\partial}{\partial r}r^2\frac{\partial\xi_1}{\partial r}\right)=-ev_0\frac{\partial v_0}{\partial r}-b\xi_0\frac{\partial\xi_0}{\partial r}. \tag{22}$$

By taking the time derivative of Eq. (21) and the divergency of Eq. (22), and subtracting Eq. (22) from Eq. (21), we obtain the equation for ξ_1 as follows:

$$\left(\frac{\omega_0}{\omega}\right)^2\frac{\partial^2\xi_1}{\partial t^2}-\frac{a}{m}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\xi_1}{\partial r}\right)+\frac{d}{m}\frac{1}{r^2}\frac{\partial}{\partial r}\left\{r^2\frac{\partial}{\partial r}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\xi_1}{\partial r}\right)\right]\right\}$$

$$=\left[\left(\frac{\omega_0}{\omega}\right)^2-1\right]\frac{\partial^2\xi_0}{\partial t^2}+g_1(r,t)\approx-2\frac{\omega_1}{\omega_0}\frac{\partial^2\xi_0}{\partial t^2}+g_1(r,t)=2\omega_1\omega_0\xi_0+g_1(r,t), \tag{23}$$

where

$$g_1(r,t)=\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\left(-f\frac{\partial(\xi_0 v_0)}{\partial t}+ev_0\frac{\partial v_0}{\partial r}+\frac{b}{m}\xi_0\frac{\partial\xi_0}{\partial r}\right)\right]. \tag{24}$$

The time dependence of function $g_1(r,t)$ can be presented as

$$g_1(r,t)=g_{10}(r)+g_{12}(r)\cos(2\omega_0 t), \tag{25}$$

where $g_{10}(r)$ and $g_{12}(r)$ are functions of r only. The expressions for $g_{10}(r)$ and $g_{12}(r)$ are presented in the Appendix. The condition that there be no resonance term [39] on the right-hand side of Eq. (23) leads to $\omega_1=0$. Note that the resonance term condition is only discussed in the time dependence case in Ref. [39]. This consideration can be extended to the case of spatial dependence (see also [13]). The finite solution of Eq. (23) at $r=0$ can be found by applying the general method of the nonhomogeneous differential equation solution [40] and has the form

$$\xi_1(r,t)=\xi_{10}(r)+\xi_{12}(r)\cos(2\omega_0 t), \tag{26}$$

where $\xi_{10}(r)$ and $\xi_{12}(r)$ are functions of r only:

$$\xi_{10}(r)=c_1\frac{\sinh(\kappa_0 r)}{\kappa_0 r}+\frac{m}{d\kappa_0^2}\left(\frac{1}{r}\int_0^r dr' g_{10}(r')r'^2-\int_0^r dr g_{10}(r')r'+\frac{\sinh(\kappa_0 r)}{\kappa_0 r}\int_0^r dr' g_{10}(r')r'\cosh(\kappa_0 r')\right.$$

$$\left.-\frac{\cosh(\kappa_0 r)}{\kappa_0 r}\int_0^r dr' g_{10}(r')r'\sinh(\kappa_0 r')\right), \tag{27}$$

$$\xi_{12}(r)=c_2\frac{\sin(k_2 r)}{k_2 r}+c_3\frac{\sinh(\kappa_2 r)}{\kappa_2 r}+\frac{m}{d(k_2^2+\kappa_2^2)}\left(-\frac{\sin(k_2 r)}{k_2 r}\int_0^r dr' g_{12}(r')r'\cos(k_2 r')+\frac{\cos(k_2 r)}{k_2 r}\int_0^r dr' g_{12}(r')r'\sin(k_2 r')\right.$$

$$\left.+\frac{\sinh(\kappa_2 r)}{\kappa_2 r}\int_0^r dr' g_{12}(r')r'\cosh(\kappa_2 r')-\frac{\cosh(\kappa_2 r)}{\kappa_2 r}\int_0^r dr' g_{12}(r')r'\sinh(\kappa_2 r')\right). \tag{28}$$

Here c_1, c_2, and c_3 are constants, $\kappa_0^2=a/d$, and the values of k_2 and κ_2 are related to the vibration frequency ω_0 by the dispersion relations

$$4\omega_0^2=\frac{a}{m}k_2^2+\frac{d}{m}k_2^4, \tag{29}$$

$$4\omega_0^2=-\frac{a}{m}\kappa_2^2+\frac{d}{m}\kappa_2^4. \tag{30}$$

Using the boundary condition (17), we can determine the constant c_1 and the combination of the constants c_2 and c_3.

By substituting Eqs. (12), (13), and Eqs. (26)–(28) into Eq. (22) we find the velocity $v_1(r,t)$ in the form

$$v_1(r,t)=v_{10}(r)t+v_{12}(r)\sin(2\omega_0 t)/(2\omega_0). \tag{31}$$

Note that

$$\frac{1}{r^2}\frac{\partial[r^2 v_{10}(r)]}{\partial r}=0,$$

and as a result of the continuity equation (21), there are no density fluctuations related with the first term in Eq. (31). Therefore there is not any influence of this term on the integral properties of the density distribution. The expressions for the radial functions $\xi_{10}(r)$, $\xi_{12}(r)$, $v_{10}(r)$, and $v_{12}(r)$ are too cumbersome and so we have not presented them.

By using the same method as before we may easily obtain the equation for $\xi_2(r,t)$:

$$\left(\frac{\omega_0}{\omega}\right)^2\frac{\partial^2\xi_2}{\partial t^2}-\frac{a}{m}\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\xi_2}{\partial r}\right)+\frac{d}{m}\frac{1}{r^2}\frac{\partial}{\partial r}\left\{r^2\frac{\partial}{\partial r}\left[\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\xi_2}{\partial r}\right)\right]\right\}$$
$$=\left[\left(\frac{\omega_0}{\omega}\right)^2-1\right]\frac{\partial^2\xi_0}{\partial t^2}+g_2(r,t)$$
$$\approx-2\frac{\omega_2}{\omega_0}\frac{\partial^2\xi_0}{\partial t^2}+g_2(r,t)=2\omega_2\omega_0\xi_0+g_2(r,t), \quad (32)$$

where

$$g_2(r,t)=\frac{1}{r^2}\frac{\partial}{\partial r}\left[r^2\left(-f\frac{\partial(\xi_1 v_0+\xi_0 v_1)}{\partial t}+e\left(v_1\frac{\partial v_0}{\partial r}+v_0\frac{\partial v_1}{\partial r}\right)+\frac{b}{m}\left(\xi_1\frac{\partial\xi_0}{\partial r}+\xi_0\frac{\partial\xi_1}{\partial r}\right)+\frac{c}{m}\xi_0^2\frac{\partial\xi_0}{\partial r}\right)\right]. \quad (33)$$

The right-hand side of Eq. (32) has a resonance term, which is proportional to $\xi_0(r,t)$. The nonresonance term condition [39] in this order of the perturbation theory leads to the finite correction of the oscillation frequency:

$$\omega_2\equiv\omega_{shift}(b,c,e,f)=\frac{\alpha^2}{8}\frac{bk_0[b+f(a+dk_0^2)]}{a\sqrt{m(a+dk_0^2)}}=\frac{\pi^4}{12}\frac{\hbar b\{b+f[a+d(\pi/r_0)^2A^{-2/3}]\}}{mar_0^2[a+d(\pi/r_0)^2A^{-2/3}]}A^{-5/3}. \quad (34)$$

The vibration frequency changes in third order of perturbation theory. This frequency shift of oscillations ω_{shift} is proportional to the square of the amplitude of vibrations; see also [39]. Note that $\omega_{shift}(b,c,e,f)=\omega_{shift}(b,f)$, because terms with c or f proportional to the resonance term $\xi_0(r,t)$ are absent on the right-hand side of Eq. (32).

The energy of the one-phonon $(1\hbar\omega)$ giant resonance related to the density vibrations in a nucleus is equal to

$$E_{1\hbar\omega}=\hbar[\omega_0+\omega_{shift}(b,f)]=\hbar(\omega_0+\omega_2). \quad (35)$$

Here we have taken into account the semiclassical quantization rule for classical periodic motion.

The N-phonon giant resonance in the nucleus is defined as a coherent excitation of N one-phonon oscillations. The density fluctuation of the N-phonon giant resonance in the case of small value of N may be presented as

$$\xi_{N\hbar\omega}(r,t)=N\xi_0(r,t)=N\alpha\sin(\omega_0 t)\frac{\sin(k_0 r)}{k_0 r}.$$

Thus, the vibration amplitude of the N-phonon excitation is N times larger than the amplitude of the one-phonon state. The N-phonon density oscillations in a nucleus are also described by the system of hydrodynamic expressions (4) and (5). The frequency shift of the N-phonon excitation due to the anharmonicity can be easily obtained from the expressions for the one-phonon case by substituting constants related to nonlinear terms in Eqs. (4) and (5):

$$b_{N\hbar\omega}\to Nb_{1\hbar\omega}, \quad c_{N\hbar\omega}\to N^2c_{1\hbar\omega}, \quad e_{N\hbar\omega}\to Ne_{1\hbar\omega}=N, \quad f_{N\hbar\omega}\to Nf_{1\hbar\omega}=N. \quad (36)$$

Therefore the energy of the N-phonon excitation is given by

$$E_{N\hbar\omega}=\hbar[N\omega_0+\omega_{shift}(Nb,N^2c,Ne,Nf)]=N\hbar(\omega_0+N\omega_2). \quad (37)$$

Let us define the ratios of energies, V_N and W_{NM}, as

$$V_N=\frac{E_{N\hbar\omega}-N\hbar\omega_0}{\hbar\omega_0}=\frac{\omega_{shift}(Nb,N^2c,Ne,Nf)}{\omega_0}=N^2\frac{\omega_2}{\omega_0},$$
$$W_{NM}=\frac{E_{N\hbar\omega}-E_{M\hbar\omega}}{E_{M\hbar\omega}}\approx\frac{N-M}{M}\left(1+N\frac{\omega_2}{\omega_0}\right). \quad (38)$$

The ratios V_N and W_{NM} quantify the nonlinearity (or anharmonicity) of the N-phonon excitation and the relative anharmonicity of the N-phonon and M-phonon states, respectively. These ratios give $V_N=0$ and $W_{NM}=(N-M)/M$ for harmonic oscillations. Deviations from these limiting values give us the quantitative characteristics of anharmonicity.

The ratio V_N is a purely theoretical quantity. The ratio W_{NM} can be derived from experimental data. Moreover, the data on the ratio W_{21} for electric giant isovector dipole and isoscalar quadrupole resonances can be extracted from experimental data [19–22]. The ratios V_1 and W_{21} will be analyzed in detail for isoscalar monopole vibrations in next section.

III. ISOSCALAR MONOPOLE EXCITATION

The proton and neutron densities vibrate in phase in the case of isoscalar oscillations. These vibrations are described by Eqs. (4) and (5) and satisfy kinematical and dynamical boundary conditions (15)–(17) on the free surface of a nucleus. By using expressions obtained in the previous section we can easily evaluate the energy of the isoscalar GMR. The resonance energy depends on the constants a_0, b_0, and d_0 of the energy density functional (3).

For symmetrical nuclear matter and the realistic Skyrme

energy density functional, the constants a_0, b_0, c_0, and d_0 are equal to

$$a_0=\rho_\infty^2\frac{\partial^2}{\partial\rho^2}\left(\frac{\mathcal{E}_{Sk}}{\rho}\bigg|_{\nabla\rho^{p(n)}=0}\right)$$
$$=-\frac{(18\pi^4)^{1/3}\hbar^2}{30m}\rho_\infty^{2/3}+\frac{t_3}{16}\alpha_0(\alpha_0+1)\rho_\infty^{\alpha_0+1}$$
$$+\frac{(18\pi^4)^{1/3}}{48}[3t_1+t_2(5+4x_2)]\rho_\infty^{5/3}, \tag{39}$$

$$b_0=\rho_\infty^3\frac{\partial^3}{\partial\rho^3}\left(\frac{\mathcal{E}_{Sk}}{\rho}\bigg|_{\nabla\rho^{p(n)}=0}\right)$$
$$=\frac{2(18\pi^4)^{1/3}\hbar^2}{45m}\rho_\infty^{2/3}+\frac{t_3}{16}\alpha_0(\alpha_0^2-1)\rho_\infty^{\alpha_0+1}$$
$$-\frac{(18\pi^4)^{1/3}}{144}[3t_1+t_2(5+4x_2)]\rho_\infty^{5/3}, \tag{40}$$

$$c_0=2\rho_\infty^4\frac{\partial^4}{\partial\rho^4}\left(\frac{\mathcal{E}_{Sk}}{\rho}\bigg|_{\nabla\rho^{p(n)}=0}\right)$$
$$=-\frac{28(18\pi^4)^{1/3}\hbar^2}{135m}\rho_\infty^{2/3}+\frac{t_3}{8}\alpha_0(\alpha_0^2-1)(\alpha_0-2)\rho_\infty^{\alpha_0+1}$$
$$+\frac{(18\pi^4)^{1/3}}{54}[3t_1+t_2(5+4x_2)]\rho_\infty^{5/3}, \tag{41}$$

$$d_0=\frac{1}{64}[9t_1-t_2(5+4x_2)]\rho_\infty. \tag{42}$$

Here we have taken into account that the proton ρ^p and neutron ρ^n densities are connected with the total density by the expression $\rho^p(r,t)=\rho^n(r,t)=\frac{1}{2}\rho_\infty[1+\xi(r,t)]$. The realistic Skyrme energy density functional is defined within the Thomas-Fermi approach as [33]

$$\mathcal{E}_{Sk}=\frac{3(9\pi^4)^{1/3}\hbar^2}{10m}[(\rho^p)^{5/3}+(\rho^n)^{5/3}]+\frac{1}{2}t_0\left[\left(1+\frac{1}{2}x_0\right)\rho^2-\left(x_0+\frac{1}{2}\right)[(\rho^p)^2+(\rho^n)^2]\right]+\frac{1}{12}t_3\rho^{\alpha_0}\left[\left(1+\frac{1}{2}x_3\right)\rho^2-\left(x_3+\frac{1}{2}\right)\right.$$
$$\left.\times[(\rho^p)^2+(\rho^n)^2]\right]+\frac{3(9\pi^4)^{1/3}}{20}\left[t_1\left(1+\frac{1}{2}x_1\right)+t_2\left(1+\frac{1}{2}x_2\right)\right][(\rho^p)^{5/3}+(\rho^n)^{5/3}]\rho+\frac{3(9\pi^4)^{2/3}}{20}\left[t_2\left(x_2+\frac{1}{2}\right)\right.$$
$$\left.-t_1\left(x_1+\frac{1}{2}\right)\right][(\rho^p)^{8/3}+(\rho^n)^{8/3}]+\frac{1}{16}\left[3t_1\left(1+\frac{1}{2}x_1\right)-t_2\left(1+\frac{1}{2}x_2\right)\right](\nabla\rho)^2-\frac{1}{16}\left[3t_1\left(x_1+\frac{1}{2}\right)+t_2\left(x_2+\frac{1}{2}\right)\right]$$
$$\times[(\nabla\rho^p)^2+(\nabla\rho^n)^2], \tag{43}$$

where $\rho=\rho^p+\rho^n$ and t_0, t_1, t_2, t_3, x_0, x_1, x_2, x_3, and α_0 are the Skyrme force parameters [32,33].

The values of the constants a_0, b_0, c_0, and d_0 and the parameters of the most successful sets of Skyrme nucleon-nucleon interactions [34–37] are shown in Table I. Note that the three-particle contribution related to α_0 in Eq. (44) has a different density dependence for these sets of parameters (see Table I). The values of the excitation energies for isoscalar density vibrations in the nucleus within the anharmonic E and harmonic $\hbar\omega_0$ approximations and the ratios V_1 and W_{21} are presented in Table II. The theoretical and experimental [10] energies and ratios in Table II are calculated for nuclei with nucleon number $A=40$, $A=90$, and $A=208$. The theoretical values are given for different parameter sets of Skyrme forces.

The values of the constants a_0, b_0, c_0, and d_0 evaluated for different sets of Skyrme forces are spread over wide intervals (see Table I). We may shorten these intervals if we know the experimental values of the ratio W_{21}. Unfortunately we do not have these data for the isoscalar GMR.

The values of excitation energies calculated in nuclei with $A=40$, 90, and 208 for different sets of Skyrme forces overestimate the experimental values of the isoscalar monopole resonance energy in ^{40}Ca, ^{90}Zr, and ^{208}Pb [10], respectively (see Table II).

The energy of the isoscalar GMR is often described by [4,6–9]

$$E=\left(\frac{\hbar^2K}{m\langle R^2\rangle}\right)^{1/2}, \tag{44}$$

where $\langle R^2\rangle$ is the mean square nuclear radius. Equation (44) differs from our expression [see Eqs. (14) and (35)] for the evaluation of the GMR energy, because we take into account the term related to d in the harmonic approximation. The value of constant $d=d_0$ is positive for all sets of Skyrme forces (see Table I). Therefore the energy of the isoscalar GMR evaluated in our approach is somewhat higher than that obtained by using Eq. (44) for the same value of K. Note

TABLE I. The parameters of Skyrme interactions and the constants of the energy density functional (3).

	Sk3 [34]	Ska [35]	SkM* [36]	RATP [37]
t_0 (MeV fm^3)	−1128.75	1602.78	−2645.00	−2160
t_1 (MeV fm^5)	395.00	570.00	410.00	513.00
t_2 (MeV fm^5)	−95.00	−67.00	135.00	121.00
t_3 (MeV fm$^{3+3\alpha_0}$)	14000.00	8000.00	15595.00	11600.00
x_0	0.45	−0.02	0.09	0.418
x_1	0.00	0.00	0.00	−0.36
x_2	0.00	0.00	0.00	−2.29
x_3	1.00	−0.286	0.00	0.586
α_0	1.00	1/3	1/6	1/5
$\rho_\infty=3/(4\pi r_0^3)$ (fm^{-3})	0.1453	0.1554	0.1603	0.1599
$a_0=K/9$ (MeV)	355.5/9	263.4/9	216.6/9	239.6/9
b_0 (MeV)	3.76	−11.1	−14.3	−13.0
c_0 (MeV)	−11.2	12.5	21.8	17.9
d_0 (MeV fm^2)	9.15	13.3	10.9	12.8

that the energy of the isoscalar GMR obtained in microscopic RPA theory for the same set of Skyrme forces is also smaller than $\hbar\omega_0$ in Table II especially in light nuclei. Such a discrepancy is connected with the approximation of the step density distribution used in our model. It is known that the GMR energy evaluated in a hydrodynamic model is reduced if the finite thickness of the diffuse layer is taken into account; for details see [40–43].

The anharmonicity is stronger in the light nuclei than in the heavy ones because the vibration amplitudes in the light nuclei are larger than in the heavy ones. As a result of anharmonicity, the excitation energies of the isoscalar one-phonon resonance, $E_{1\hbar\omega}=\hbar\omega_0(1+V_1)$, rise by 2% for heavy and by $\approx 14-18\,\%$ for light nuclei compared to the harmonic approach. We recall that the harmonic limit values for the ratios (39) and (40) are $V_1=0$ and $W_{21}=1$. Different sets of Skyrme interactions give various values for the ratios V_1 and W_{21}. The effect of the density vibration's anharmonicity for the Sk3 set is largest compared to the other sets. This is because $V_N\propto b^2=(3a_0+b_0)^2$, and the value of the parameter b_0 is positive in the case of Sk3 forces and is negative for Ska, Skm*, and RATP parameter sets. The value of compressibility $K=9a_0$ is the largest for the Sk3 parameter set, as we see in Table I.

TABLE II. The giant resonance energies evaluated in nonlinear E and linear $\hbar\omega_0$ approaches, and the ratios V_1 and W_{21} for different sets of Skyrme forces in our hydrodynamical model and in QTDHF, RMF, and ERPA models. The energy values, amplitudes, and ratios V_1 and W_{21} are given for nuclei with nucleon numbers $A=40$, 90, and 208. The anharmonicity effects of the two strongest monopole states in ^{208}Pb are studied in the ERPA. The quantities for the highest states obtained in the ERPA are given in parentheses.

	Sk3 [34]	Ska [35]	SkM* [36]	RATP [37]	RMF [17]	QTDHF [15]	HFB+RPA [8]	ERPA [18]	Expt. [10]
					$A=40$				
E (MeV)	39.60	35.77	32.71	34.62	19.0	21.0	16.2		18.9±0.4
$\hbar\omega_0$ (MeV)	33.66	31.47	28.91	30.58	21.5	26.1	21.3		
V_1	0.18	0.14	0.13	0.13	−0.12	−0.20	−0.24		
W_{21}	1.35	1.27	1.26	1.26					
					$A=90$				
E (MeV)	26.61	24.06	22.04	23.27	22.7				16.44±0.07
$\hbar\omega_0$ (MeV)	25.02	22.87	20.99	22.15	23.3				
V_1	0.06	0.05	0.05	0.05	−0.026				
W_{21}	1.13	1.10	1.10	1.10					
					$A=208$				
E (MeV)	19.02	17.07	15.64	16.49	21.0		13.0	13.4 (14.78)	13.91±0.11
$\hbar\omega_0$ (MeV)	18.61	16.76	15.37	16.20	21.1		13.5	13.6(15.0)	
V_1	0.022	0.018	0.018	0.018	−0.005		−0.037	−0.015(−0.015)	
W_{21}	1.044	1.036	1.035	1.036				1.054(1.041)	

The ratios V_N and W_{MN} are a quadratic function of $b=3a_0+b_0$ [see Eq. (34)]. The ratio $V_N(A)$ is proportional to $A^{-4/3}$ for large values of A, because $\omega_0 \propto A^{-1/3}$ and $\omega_2 \propto A^{-5/3}$. The same A dependence, sign, and similar amplitude of $V_2(A)$ are also obtained in [26]. Note that the nature of anharmonicity in our paper is the same as in Ref. [26]. Another A dependence of V_2 as $A^{-2/3}$ is found in Ref. [25], which is related to the Pauli principle correction [24,25]. Note that $V_2 \propto A^{-2/3}$ in the slab of nuclear matter [13]. The different A dependences of V_2 obtained within similar approaches for the slab and spherical space distributions of nuclear matter are related to the different geometries of the nuclear density distributions. Note that vibrations in the slab and spherical nuclei are described by different equations [compare Eqs. (4) and (5) in this paper and in [13]], and have, therefore, different A dependences of ω_2.

The anharmonicity effects for the isoscalar GMR are estimated in the QTDHF [15], RMF [17], HFB+RPA [8] and ERPA [18] approximations. The values of E, $\hbar\omega_0$, and V_1 obtained in these approximations are given in Table II. Our values of V_1 obtained for Sk3, Ska, SkM*, and RATP sets of Skyrme forces are smaller than QTDHF and HFB+RPA predictions and larger than RMF result.

The dependence of the Hartree-Fock deformation energy on the value of the mean square nuclear radius $\langle R^2\rangle$ is evaluated by using the constraint method in Refs. [15,17]. It is shown in [15,17] that the dependence of the Hartree-Fock energy on $\langle R^2\rangle$ differs from that in the harmonic approximation. This induces anharmonicity effects in Refs. [15,17]. Note that the radial dependence of the transition density [the difference between dynamic $\rho(r,t)$ and static ρ_∞ densities] for the isoscalar monopole resonance (12) is not directly and uniquely related to the variation of the mean square radius.

The nonlinearity connected with pairing effects is studied in Ref. [8] within the constrained Hartree-Fock-Bogoliubov method. It is shown in Ref. [8] that the pairing interaction gives an additional strong contribution to the dependence of the Hartree-Fock-Bogoliubov energy on $\langle R^2\rangle$ in ^{40}Ca. The coupling between phonons of different natures and multipolarities is taken into account in Ref. [18].

The anharmonicity shifts of the resonance energy are higher in our hydrodynamical approach in contrast to the opposite effects in the microscopic QTDHF, HFB+RPA, ERPA, and RMF theories. The nonlinearity obtained in our approach is related to the bulk (volume) property of isoscalar giant monopole vibrations due to nonlinear terms in Eqs. (4) and (5). However, nonlinear effects of other natures are considered in Refs. [8,15,17,18]. Note that the finite thickness of the nuclear diffuse surface, pairing effects, the coupling between phonons of different natures, and other effects are taken into account in microscopic approaches [8,15–18]. These additional effects may cause the differences between various model predictions.

The energy of the two-phonon isoscalar GMR is also changed due to anharmonicity, since the ratio W_{21} deviates from the harmonic limit as we see in Table II. The values of W_{21} obtained for Ska, SkM*, and RATP sets of Skyrme forces are similar. The values of anharmonicity for the two-phonon monopole resonance obtained in [26] are close to our values. The nonlinearity effect on V_1 an W_{21} is the largest for the Sk3 parameter set. The anharmonicity effect obtained in ERPA theory is smaller than in our approach for the two-phonon monopole excitations (see the value W_{21} in Table II).

Note that we consider anharmonicity by using perturbation theory. Therefore our model lost accuracy in the case of large anharmonicity. As a result of this, it is necessary to consider the anharmonicity of the N-phonon state numerically at large values of N.

IV. CONCLUSION

The anharmonic terms of the energy density functional enlarged considerably the energy of the one- and two-phonon excitations. Therefore the isoscalar GMR should be studied in the framework of nonlinear models especially for light nuclei. However, the anharmonicity of the two-phonon giant resonance should be taken into account even in heavy nuclei.

Nonlinear effects are very important for the extraction of the nuclear compressibility from the A dependence of GMR experimental energies, because the anharmonicity changes energies of giant resonances.

The experimental study of the N-phonon giant resonances of different multipolarities gives us information about both the value and the A dependence of the ratio W_{NM}. The additional experimental information may help to select sets of Skyrme forces and to reduce the variations of the constants a, b, and d. Note that only limited experimental information about one-phonon and two-phonon isovector giant dipole resonances is available for analysis [19,20].

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APPENDIX

The expressions for $g_{10}(r)$ and $g_{12}(r)$ are obtained by substituting (12) and (13) to (22)

$$\begin{aligned} g_{10}(r) = &-\alpha^2[-bk^4r^2+2bk^5r^3\sin(2kr)-6ew^2m+bk^4r^2\cos(2kr)-2bk^6r^4\cos(2kr)+6ew^2m\cos(2kr) \\ &-6ew^2mk^3r^3\sin(2kr)-11ew^2mk^2r^2\cos(2kr)+2ew^2mk^4r^4\cos(2kr)+12ew^2mkr\sin(2kr) \\ &-ew^2mk^2r^2]/(4k^6r^6m), \end{aligned} \tag{A1}$$

$$g_{12}(r) = -\alpha^2[-6ew^2m + bk^4r^2 - 2bk^5r^3\sin(2kr) + 2bk^6r^4\cos(2kr) - bk^4r^2\cos(2kr) - 2fw^2k^2r^2m\cos(2kr) - 4fw^2k^3r^3m\sin(2kr) - ew^2mk^2r^2 + 2ew^2mk^4r^4\cos(2kr) + 6ew^2m\cos(2kr) + 4fw^2k^4r^4m\cos(2kr) - 6ew^2mk^3r^3\sin(2kr) + 2fw^2k^2r^2m - 11ew^2mk^2r^2\cos(2kr) + 12ew^2mkr\sin(2kr)]/(4k^6r^6m). \quad \text{(A2)}$$

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